Limiting Distribution of Quadratic Chaos on Graphs[1]

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[1] Joint work with Bhaswar B. Bhattacharya, Sumit Mukherjee

Outline

Framework, Goal and Some Connections

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- Various graph based nonparametric two-sample tests in Statistics are functions of T_n. In particular, the Bahadur slopes of these statistics can be computed using a large deviation theory for T_n proved by us.

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Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli$\left(\frac{\lambda}{n}\right)$.

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- Dense Stochastic Block Models: $\mathrm{SBM}(n_1, n - n_1, p, q)$: If $n_1/n \to \alpha$, then

$$T_n \xrightarrow{D} \mathrm{Binomial}\left(\binom{N_1}{2}, p\right) + \mathrm{Binomial}\left(\binom{N_2}{2}, p\right) + \mathrm{Binomial}\left(N_1 N_2, q\right) ,$$

where $N_1 \sim \mathrm{Poisson}(\alpha\lambda)$, $N_2 \sim \mathrm{Poisson}((1-\alpha)\lambda)$ are independent, and the three summands above are independent, given N_1, N_2.

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- Disjoint Union of n many $n-$stars: (A Sparse Graph Example):
  $T_n \xrightarrow{D} \mathrm{Poisson}(N)$, where $N \sim \mathrm{Poisson}(\lambda)$.

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Assumptions:

- Stretched Graphon associated with G_n: $W_n : [0, np_n]^2 \mapsto [0, 1]$ given by:

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- W_n converges in cut norm to a symmetric, integrable function $W : [0,\infty)^2 \mapsto [0,1]$, i.e. for all $K \in \mathbb{N}$ large enough,

$$\sup_{f,g:[0,K]\mapsto[-1,1]} \left| \int_{[0,K]^2} \left(W_n(x,y) - W(x,y)\right) f(x)g(y)dxdy \right| \to 0 \; .$$

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- There exists an integrable function $d : [0,\infty) \mapsto [0,\infty)$, such that for all $K \in \mathbb{N}$ large enough,

$$(U_K, d_n(U_K)) \xrightarrow{D} (U_K, d(U_K)) \ ,$$

where U_K is a uniform random variable on the interval $[0, K]$.

Main Result:

Under the assumptions in the previous slide, $T_n \xrightarrow{D} Q_1 + Q_2 + Q_3$, where

- $Q_3 \sim \text{Poisson}(\lambda)$, and is independent of (Q_1, Q_2) ,
- Q_2 conditional on some random variable R_2 has a Poisson(R_2) distribution,
- The joint moment generating function of Q_1 and R_2 is given by:

$$\mathbb{E}e^{-sQ_1 - tR_2} = \mathbb{E}\exp\left\{\frac{1}{2}\int_0^\infty \int_0^\infty \phi_{W,s}(x,y)dN(x)dN(y) - t\int_0^\infty \Delta(x)dN(x)\right\}$$

where

1. $\phi_{W,s}(x,y) = \log\left(1 - W(x,y) + W(x,y)e^{-s}\right)$,
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Dense Graphs and a Converse

If G_n is a sequence of dense graphs converging in cut distance to a graphon W, then $Q_2 = Q_3 = 0$ and $\Delta \equiv 0$. Conversely, for any sequence of dense graphs with $p_n^2|E(G_n)| = O(1)$, the limiting distribution of T_n, if exists, must be of the above form for some symmetric, measurable, integrable $W : [0,\infty)^2 \mapsto [0,1]$.

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- It holds when $X_1, X_2, \ldots$ are i.i.d. non-negative integer valued random variables with $p_n := \mathbb{P}(X_1 = 1) \to 0$, such that $|\mathbb{E}(G_n)|p_n^2 = \Theta(1)$ and $\lim_{n\to\infty} \frac{\mathbb{E}X_1}{p_n} = 1$.

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- Sparse Hypergeometric: $\mathrm{HG}(N_n, K_n, m_n)$, where $m_n K_n / N_n \to 0$. (Number of successes in m_n draws WOR from a population of size N_n having exactly K_n success states.)

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Answer: By Partitioning the Graph

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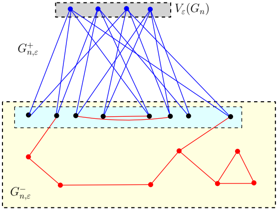

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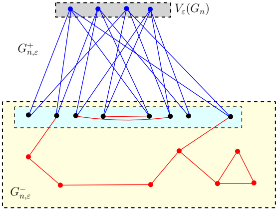

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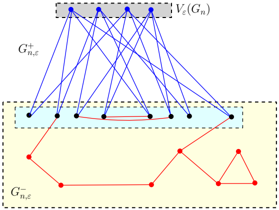

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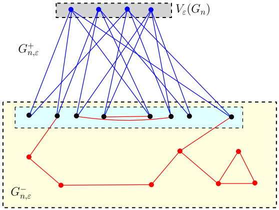

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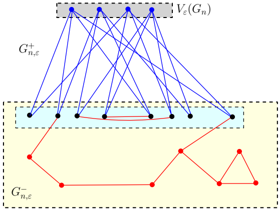

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- (Q_1, Q_2) appears as the limit of $(T_{n,1}, T_{n,2})$, by an edge-independent random graph planting argument.

Outline

References

1. C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi, Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, *Advances in Mathematics*, Vol. 219, 1801–1851, 2009.

2. C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi, Convergent sequences of dense graphs II. Multiway cuts and statistical physics, *Annals of Mathematics*, Vol. 176, 151–219, 2012.

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4. B. B. Bhattacharya, S. Mukherjee, and S. Mukherjee, Birthday paradox, monochromatic subgraphs, and the second moment phenomenon, `arXiv:1711.01465`, 2017.